- **Proof of (3)** $t(R)=R\cup R^2\cup R^3\cup\ldots$
  - Consider arbitrary pairs **<*x*,*y*> and <*y*,*z*>**

$$<x,y>\in R\cup R^2\cup R^3\cup\ldots.\ \wedge\ <y,z>\in R\cup R^2\cup R^3\cup\ldots.$$

$$\Rightarrow\ <x,z>\in R\cup R^2\cup R^3\cup\ldots.$$

Therefore, by the transitivity of $R\cup R^2\cup R^3\cup\ldots$. **We have**

$$t(R)\subseteq R\cup R^2\cup R^3\cup\ldots$$

  - Next, we prove by induction that $R^n\subseteq t(R)$.

*For* $n=1$, the statement is obviously true. Assume it holds for $n=k$.

***For any*** **<*x*,*y*>, we have**

$$<x,y>\in R^{k+1}\Rightarrow\ <x,y>\in R^k\circ R\ \Rightarrow\exists t\ (<x,t>\in R^k\wedge<t,y>\in R)$$

$$\Rightarrow\exists t\ (<x,t>\in t(R)\wedge<t,y>\in t(R))\ \Rightarrow<x,y>\in t(R)\quad (t(R)\text{ transitive})$$

Thus, $R\cup R^2\cup R^3\cup\ldots\subseteq t(R)$

- Let the relation matrices of $R$, $r(R)$, $s(R)$, $t(R)$ be $M$, $M_r$, $M_s$ and $M_t$, respectively. Then, we have:

$$M_r = M + E$$

$$M_s = M + M'$$

$$M_t = M + M^2 + M^3 + \ldots$$

- where $E$ is the identity matrix of the same order as $M$, and $M'$ is the transpose of $M$.

**Note:** In the above equations, the matrix elements are added using logical addition.

## ↳ Closure Operations on Relation Graphs



- Let the relation graphs of $R$, $r(R)$, $s(R)$, $t(R)$ be denoted by $G$, $G_r$, $G_s$, $G_t$, respectively.

  **Then**, the vertex sets of $G_r$, $G_s$, $G_t$ are the same as the vertex set of $G$.

- In addition to the edges of $G$ , new edges are added in the following ways：
  - For each vertex in $G$, if there is no cycle, add a cycle. The resulting graph is $G_r$.
  - For each directed edge $x_i \to x_j$, (with $i \neq j$), add a reverse edge $x_j \to x_i$ . The resulting graph is $G_s$.
  - For each vertex $x_i$ in $G$, examine all paths starting from $x_i$ ， if there is no edge from $x_i$ to any node $x_j$ in the path, add the corresponding edge. After checking all vertices, the resulting graph is $G_t$ .

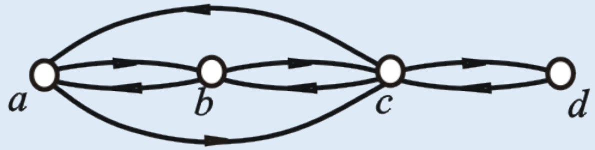
# 4.3.2 Closure of Relations

## ↳ Closure Operations on Relation Graphs (e.g.)

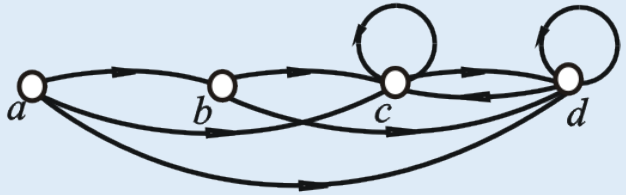
e.g. **Example:** Let $A=\{a,b,c,d\}$, $R=\{<a,b>,<a,c>,<b,c>,<c,d>,<d,c>\}$, $R$ and $r(R)$, $s(R)$, $t(R)$ the relation graph is shown.


$R$


$r(R)$




$s(R)$




$t(R)$

- **Algorithm Idea:** Consider a sequence of matrices $M_0$, $M_1$, …, $M_n$ of size $n+1$, where the element in the i-th row and j-th column of matrix $M_k$ is denoted as $M_k[i,j]$. For k=0,1,…,n, $M_k[i,j]=1$ if and only if there exists a path from $x_i$ to $x_j$ in the relation graph of $R$ , and this path passes through only the vertices in $\{x_1, x_2, …, x_k\}$ except for the endpoints. It is easy to prove that $M_0$ is the relation matrix of $R$ , and $M_n$ corresponds to the transitive closure of $R$ .
- **Warshall Algorithm:** Starting from $M_0$, calculate $M_1$, $M_2$, …, until $M_n$ .

  **From $M_k[i, j]$ *to compute* $M_{k+1}[i, j]$: $i, j \in V$.**

  **The vertex set $V_1=\{1,2, …, k\}$, $V_2=\{k+2, …, n\}$, $V=V_1\cup\{k+1\}\cup V_2$,**

  **$M_{k+1}[i,j]=1 \Leftrightarrow$ There exists a path *i to j* .**

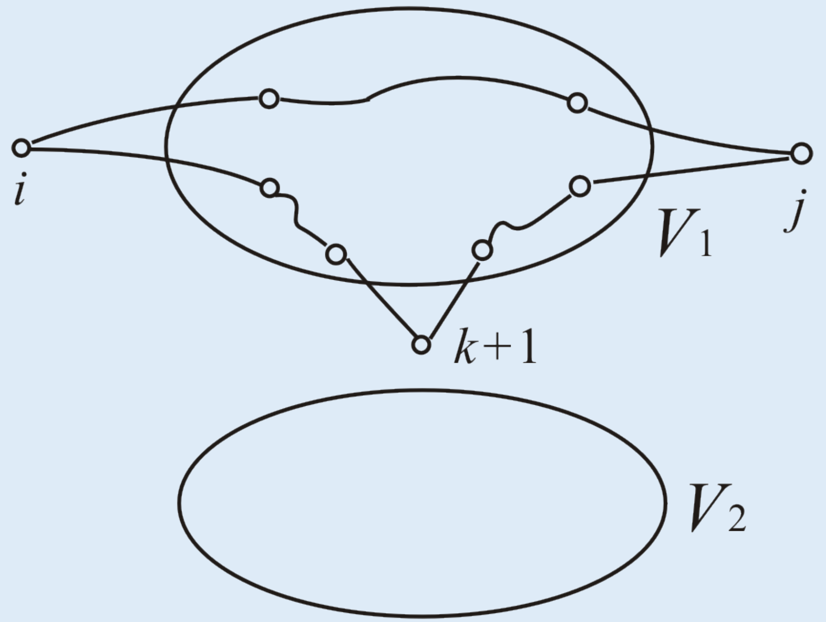
  that only passes through the points in $V_1\cup\{k+1\}$.

- These paths are divided into two categories:
  - **Category 1:** Paths that only pass through the points in $V_1$
  - **Category 2:** Paths that pass through point $k+1$

  For **Category 1** paths: $M_k[i,j]=1$

  For **Category 2** paths:

  $$M_k[i,k+1]=1 \wedge M_k[k+1,j]=1$$



- **Algorithm 4.1: Warshall** Algorithm

Input： $M$ （relation matrix of $R$ ）

Output： $M_t$ （relation matrix of $t(R)$）

1. $M_t \leftarrow M$
2. for $k \leftarrow 1$ to $n$ do
3. for $i \leftarrow 1$ to $n$ do
4. for $j \leftarrow 1$ to $n$ do
5. $M_t[i, j] \leftarrow M_t[i, j]$ or $M_t[i, k] \cdot M_t[k, j]$

**Time Complexity:** $T(n)=O(n^3)$

**Objective：**

**Key Concepts：**

# Discrete Mathematics 2025 Spring

同济经管
TONGJI SEM

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- 4.4.1 Equivalence Relations
- 4.4.2 Equivalence Classes and Quotient Sets
- 4.4.3 Partition of a Set
- 4.4.4 Partial Order Relations
- 4.4.5 Partially Ordered Sets and Hasse Diagrams

- **Definition 4.18: let** ***R*** be a relation on a non-empty set. **If** ***R*** is reflexive, symmetric, and transitive, then ***R*** is called an ***equivalence relation*** on *A*. If ***R*** is an equivalence relation and **$\langle x,y\rangle \in R$, we say that** ***x*** is equivalent to ***y***, **denoted as** **$x \sim y$.**

e.g. »» **Example:** Verify that ***R*** is an equivalence relation on *A*.

**Let** $A=\{1, 2, \ldots, 8\}$ , and define the relation ***R on A as follows*** **:**

$R=\{\langle x,y\rangle \mid x,y\in A \wedge x\equiv y \pmod 3\}$

**where** **$x\equiv y \pmod 3$** means that ***x and y*** are congruent modulo 3, i.e., the remainder when **x** is divided by **3** is equal to the remainder when ***y*** is divided by **3**.

e.g. >>>**Example: Let** $A=\{1, 2, \ldots, 8\}$ , and define the relation ***R on A as follows*** : $R=\{<x,y> \mid x,y \in A \wedge x \equiv y \pmod 3\}$

**where** $x \equiv y \pmod 3$ means that ***x and y*** are congruent modulo 3, i.e., the remainder when **x** is divided by **3** is equal to the remainder when ***y*** is divided by 3.

It is easy to verify that ***R*** is an equivalence relation on ***A***, because:

$\forall x \in A$, if $x \equiv x \pmod 3$ （reflexivity）

$\forall x,y \in A$, if $x \equiv y \pmod 3$, then $y \equiv x \pmod 3$ （symmetry）

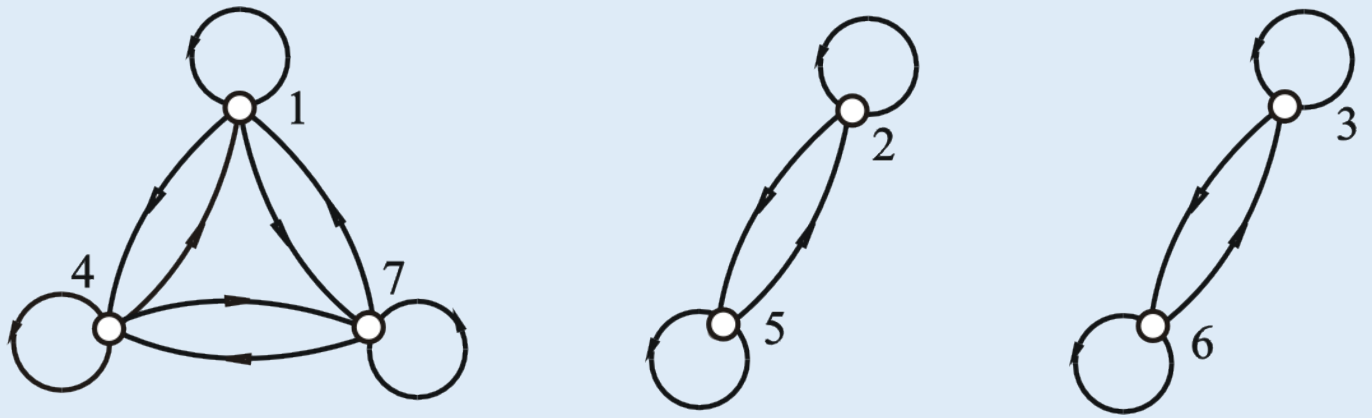
$\forall x,y,z \in A$, if $x \equiv y \pmod 3$, $y \equiv z \pmod 3$, then $x \equiv z \pmod 3$ （transitivity）

- Relation Graph of the Modulo 3 Equivalence Relation on A

  Let $A=\{1, 2, \ldots, 7\}$,

  $R=\{ \langle x,y \rangle \mid x,y \in A \land x \equiv y \pmod 3 \}$

  The relation graph of $R$ is shown below：



# 4.4 Equivalence Relations and Partial Order Relations

- **Definition 4.19: let** $R$ be an equivalence relation on a non-empty set $A$, $\forall x \in A$, define $[x]_R = \{\, y \mid y \in A \wedge xRy \,\}$
  We call $[x]_R$ the ***equivalence class*** of ***x under R***, or simply the equivalence class of $x$, denoted as $[x]$.

**Note:** $[x]_R$ is the set of all elements in $A$ that are equivalent to $x$ under the relation $R$.

$$[x]_R = \{y \in A \mid (x,y) \in R\}$$

e.g. **Example:** Equivalence classes of the modulo 3 equivalence relation on A={1, 2, … , 8}：

**[1]=[4]=[7]={1,4,7}**

**[2]=[5]=[8]={2,5,8}**

**[3]=[6]={3,6}**

- The three equivalence classes with remainders 1, 2, and 0 are disjoint, and their union is *A*.

- **Theorem 4.8:** Partition Theorem of Equivalence Classes.

  Let ***R*** be an equivalence relation on a non-empty set ***A***

  The following conclusions hold:

(1) $\forall x \in A$, $[x]$ is a non-empty subset of $A$.

(2) $\forall x, y \in A$, if $xRy$, then $[x]=[y]$.

(3) $\forall x, y \in A$, if $x \not R y$, then $[x]$ and $[y]$ are disjoint.

(4) $\bigcup_{x\in A}[x] = A$, the union of all equivalence classes is equal to $A$.

- **Theorem 4.8(1):** $\forall x \in A$, $[x]$ is a non-empty subset of $A$.

**Proof:** From the definition of equivalence classes, $\forall x \in A$, we have $[x] \subseteq A$. By reflexivity, $xRx$， so $x \in [x]$, which implies that $[x]$ is non-empty. Since all elements in the equivalence class $[x]$ are selected from the set $A$， it follows that $[x]$ is a subset of $A$.

- **Theorem 4.8(2):** $\forall x, y \in A$, if $xRy$, then $[x]=[y]$.

**Proof:** For any element a in $[x]$， $(x,a) \in R$. Since $(x,y) \in R$ and R is transitive, we can conclude that $(y,a) \in R$ , so $a \in [y]$. This proves that $[x] \subseteq [y]$.

Similarly, we can prove that $[y] \subseteq [x]$. Therefore $[x]=[y]$ .

- **Theorem 4.8(3):** $\forall x, y \in A$, if $x \not R y$, then $[x]$ and $[y]$ are disjoint.

  **Proof:**

  Suppose $[x] \cap [y] \neq \varnothing$, then there exists an element $z \in [x] \cap [y]$, which implies $z \in [x] \wedge z \in [y]$, **then** $<x,z> \in R \wedge <y,z> \in R$ ***holds***.

  By the **transitivity** and **symmetry** of $R$, ***then*** $<x,y> \in R$,

  **contradicted to** $x \not R y$.

- **Theorem 4.8(4):**

  $\bigcup_{x\in A}[x] = A$, the union of all equivalence classes is $A$.

- **Proof1:** $\cup_{x\in A}[x] \in A$ , For any $y$.

  $y \in \cup_{x\in A}[x] \Leftrightarrow \exists x\ (x\in A \wedge y\in[x])$

  $\Rightarrow y\in[x] \wedge [x]\subseteq A \Rightarrow y\in A$

  *Then* $\cup_{x\in A}[x] \in A)$

- **Proof2:** $A \in \cup_{x\in A}[x]$ , For any $y$.

  $y\in A \Rightarrow y\in[y] \wedge y\in A \Rightarrow y\in \cup_{x\in A}[x]$

  Hence $A \subseteq \cup_{x\in A}[x]$ host.

- Thus, we conclude that: $\cup_{x\in A}[x] = A$.

- **Definition 4.20:** Let $R$ be an equivalence relation on a non-empty set $A$. The set of equivalence classes of $R$ is called the ***quotient set*** of $A$ with respect to $R$, denoted by $A/R$,

$$A/R = \{ [x]_R \mid x \in A \}$$

e.g.»» **Example:** *Let* $A=\{1, 2, \ldots, 8\}$, the quotient set of $A$ with respect to the equivalence relation $R$ modulo 3 is: $A/R = \{ \{1, 4, 7\}, \{2, 5, 8\}, \{3, 6\} \}$

- The quotient sets of $A$ with respect to the identity relation and the universal relation are:

$$A/I_A = \{ \{1\},\{2\}, \ldots ,\{8\}\}$$

$$A/E_A = \{ \{1, 2, \ldots ,8\} \}$$

- **Definition 4.21:** Let $A$ be a non-empty set, $A$ family of subset $\pi$ ($\pi \subseteq P(A)$), if it satisfies the following conditions:

  (1) $\varnothing \notin \pi$; (2) $\forall x \forall y\ (x,y \in \pi \wedge x \neq y \rightarrow x \cap y = \varnothing)$; (3) $\cup \pi = A$

  Then $\pi$ is called a ***partition of A***, and the element of $\pi$ are called blocks of the partition of $A$.

e.g. **Example:** Let $A = \{a, b, c, d\}$, given the partitions:

$$\pi_1 = \{\{a, b, c\}, \{d\}\}, \qquad \pi_2 = \{\{a, b\}, \{c\}, \{d\}\}$$

$$\pi_3 = \{\{a\}, \{a, b, c, d\}\}, \qquad \pi_4 = \{\{a, b\}, \{c\}\}$$

$$\pi_5 = \{\varnothing, \{a, b\}, \{c, d\}\}, \qquad \pi_6 = \{\{a, \{a\}\}, \{b, c, d\}\}$$

**Then $\pi_1$, $\pi_2$ *are partitions of A*, while the others are not.**

- The quotient set *A/R is a partition of A* .
- Different quotient sets correspond to different partitions.
- Given any partition $\pi$ of A , we define a relation ***R on A*** as follows:

$$R=\{<x,y> \mid x,y\in A \text{ and } [x]_\pi=[y]_\pi\}$$

($[x]_\pi=[y]_\pi$ : ***x*** and ***y*** are in the same partition block of $\pi$ )

Then, ***R*** is an equivalence relation on ***A***, and the quotient set determined by this equivalence relation is exactly $\pi$.

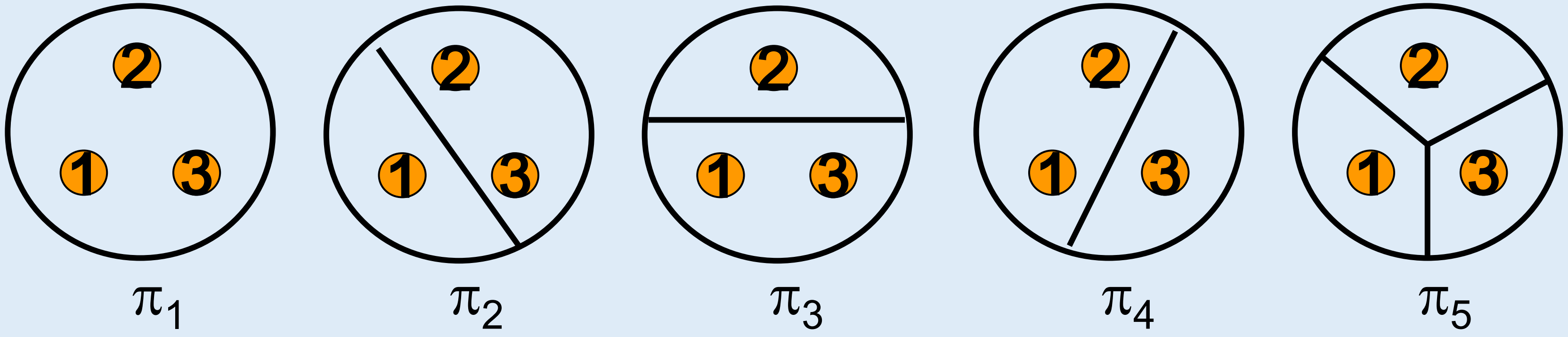
e.g.>>> **Example:** List all equivalence relations on $A=\{1,2,3\}$

**Solution approach:**

First, determine all partitions of *A*, and then write out the corresponding equivalence relations based on these partitions.

## ↳ Set Partitions and Equivalence Relation(e.g.)

Determine all partitions of $A=\{1,2,3\}$




$\boldsymbol{\pi_1}$ corresponds to the universal relation $\boldsymbol{E_A}$

$\boldsymbol{\pi_5}$ corresponds to the identity relation $\boldsymbol{I_A}$

$\boldsymbol{\pi_2,\pi_3}$ $\boldsymbol{\pi_3}$correspond to the equivalence relations $\boldsymbol{R_2}$, $\boldsymbol{R_3}$ 和 $\boldsymbol{R_4}$.

$R_2=\{<2,3>,<3,2>\}\cup I_A$

$R_3=\{<1,3>,<3,1>\}\cup I_A$

$R_4=\{<1,2>,<2,1>\}\cup I_A$

e.g. **Example:** Let $A=\{1,2,3,4\}$

- Define a binary relation $R$ on $A\times A$:

  $<<x, y>,<u, v>>\in R \Leftrightarrow x+y = u+v$, find the partition induced by $R$.

  **Solution:** $A\times A=\{<1,1>, <1,2>, <1,3>, <1,4>, <2,1>, <2,2>, <2,3>, <2,4>,<3,1>, <3,2>, <3,3>, <3,4>, <4,1>, <4,2>,<4,3>, <4,4>\}$

- According to the sum condition $<x,y>$ and $x+y=2,3,4,5,6,7,8$ which partition $A\times A$ into 7 equivalence classes:

$$(A\times A)/R=\{\{<1,1>\}, \{<1,2>,<2,1>\}, \{<1,3>, <2,2>, <3,1>\},$$
$$\{<1,4>, <2,3>, <3,2>, <4,1>\}, \{<2,4>, <3,3>, <4,2>\},$$
$$\{<3,4>, <4,3>\}, \{<4,4>\}\}$$

■ **Definition 4.22:**

A relation on a non-empty set that is **reflexive, antisymmetric, and transitive** is called a ***partial order relation*** on *A*, denoted ***by*** **≼. If <*x*, *y*>∈≼**, then we write it as ***x*≼*y***, which is read as ***x*** "less than or equal" ***y*.**

e.g. **Examples:**

- The **identity relation $I_A$ on set** *A* is a partial order relation on *A* .
- The **less than or equal to** relation, **divisibility** relation, and **subset inclusion** relation are also partial order relations on their respective sets.

## ■ Definition 4.23: Comparability

Let $R$ be a **partial order relation** on a non-empty set $A$,

- **For $x$, $y \in A$**, we say that ***x and y*** are ***comparable*** if and only if $x \preceq y$ **or** $y \preceq x$.
- **Condition for incomparability**: If no partial order relation relates $x$ and $y$, then they are **not comparable**.

## ■ Definition 4.24: Total Order

- If $R$ is a partial order on a non-empty set $A$, $\forall x, y \in A$, ***x and y*** are always comparable, $R$ is called a ***total order***.

e.g. **Examples:**

- The "**less than or equal to**" relation on numerical sets (such as real numbers and integers) is a **total order**.’
- The **divisibility relation** is **not a total order** on the set of positive integers.

■ **Definition 4.25:** Covering

$x,y \in A$, **if** $x \prec y$ and there is no $z \in A$ ***such that*** $x \prec z \prec y$, then we say that ***y covers x.***

- **Such as:** On the set **{1, 2, 4, 6}** with the divisibility relation:

**2 covers1, 4 and 6 covers2.**

**4 doesn’t cover 1.**

- **Definition 4.26: Partially ordered set.**

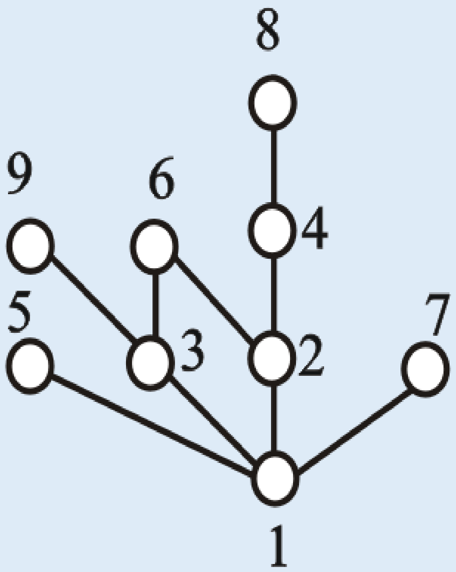
  A ***partially ordered set*** **(poset)** consists of a set $A$ together with a partial order relation $\preccurlyeq$ , denoted as **<*A*,≼>.**

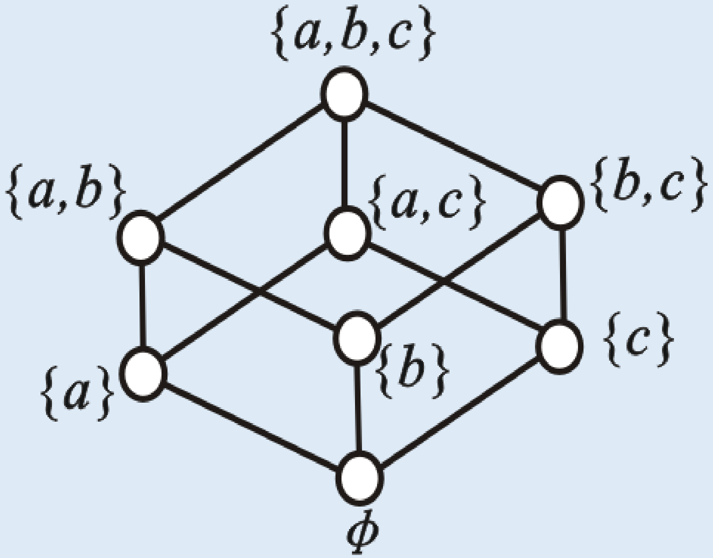
- **Such as:**
  - The set of integers with the "less than or equal to" relation forms a poset **<Z,≤>.**
  - The power set $P(A)$ with the subset inclusion relation forms a poset $<P(A),R_{\subseteq}>$.

- **Hesse Diagram:** A simplified graphical representation of a partial order that eliminates reflexivity, antisymmetry, and transitivity in the diagram.
- **Characteristics:**
  - Each node has no self-loops.
  - The order between two connected nodes is represented by their relative position: **Lower-positioned elements come earlier in the order**.
  - There is an edge between two nodes **if and only if** they have a covering relation.
- A Hasse diagram is a special type of relational graph for posets, with transitive edges removed and implicit direction.

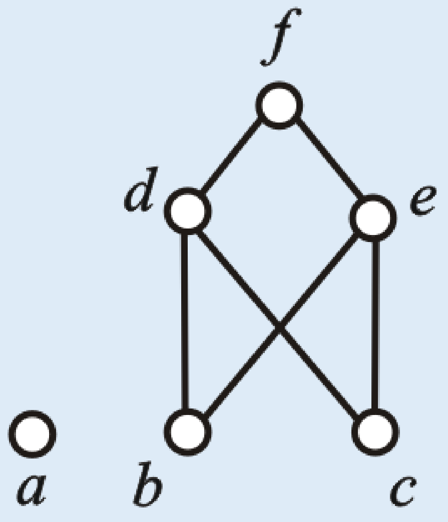
e.g. Example : <{ 1, 2, 3, 4, 5, 6, 7, 8, 9 }, $R_{Div}$>

<$P(\{a, b, c\})$, $R_{\subseteq}$>



e.g. **Example** : Given the Hasse diagram of the **partially ordered set** $<A,R>$ shown below, find the expression for the **set** $A$ **and the relation** $R$.




- **Solution:** ①Identify the Elements in $A$;②Extract the Covering Relations;③Complete transitive and reflexive relations.

$A=\{a, b, c, d, e, f\}$

$R=\{<b,d>,<b,e>,<b,f>,<c,d>,<c,e>,<c,f>,<d,f>,<e,f>\}\cup I_A$

- **Definition 4.27:** Let $<A,\preccurlyeq>$ be a **partially ordered set (poset)** , $B \subseteq A$, $y \in B$.

(1) **if** $\forall x(x \in B \rightarrow y \preccurlyeq x)$, then y is called the ***least element*** of $B$.

(2) **if** $\forall x(x \in B \rightarrow x \preccurlyeq y)$, then y is called the ***greatest element*** of $B$.

(3) **If** $\forall x(x \in B \wedge x \preccurlyeq y \rightarrow x = y)$, then y is called a ***minimal element*** of $B$.

(4) **If** $\forall x(x \in B \wedge y \preccurlyeq x \rightarrow x = y)$, y is called a ***maximal element of*** $B$.

- **Properties:**
  - In a **finite set**, minimal and maximal elements **always exist** and **may not be unique.**
  - **Least and greatest elements are not guaranteed to exist**, but if they do, they are unique.
  - The **least element** is always a **minimal element.**
  - The **greatest element** is always a **maximal element.**
  - **Isolated nodes** are **both minimal and maximal elements.**

## ↳ Upper/lower bounds, supremum, infimum of a poset



- **Definition 4.28: Let** $<A, \preccurlyeq>$ be a **partially ordered set (poset)**, and let $B \subseteq A$, $y \in A$.

(1) **If** $\forall x\ (x \in B \rightarrow x \preccurlyeq y)$ , then $y$ is called an **upper bound** of $B$.

(2) **If** $\forall x\ (x \in B \rightarrow y \preccurlyeq x)$ , then $y$ is called a **lower bound** of $B$.

(3) **Let** $C = \{y \mid y$ is an upper bound of $B\}$, the **least element** of $C$, if it exists, is called the **least upper bound (supremum) of** $B$ or the **supremum**.

(4) **Let** $D = \{y \mid y$ is a lower bound of $B\}$, The **greatest element** of $D$, if it exists, is called the **greatest lower bound (infimum) of** $B$ or the **infimum**.

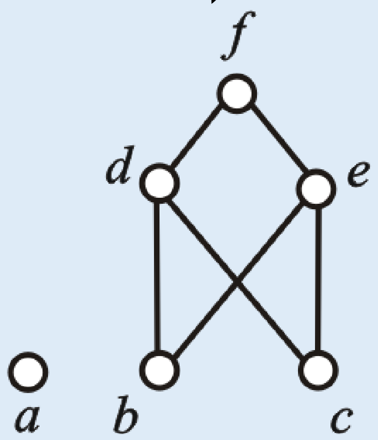
■ **Properties:**

- Lower bounds, upper bounds, infimum, and supremum are **not always guaranteed to exist.**
- Lower bounds and upper bounds, if they exist, **may not be unique**.
- The infimum and supremum, if they exist, are unique.
- The least element of a set is its infimum, and the greatest element is its supremum, however, the **reverse is not always true**.

e.g. **Example:**

(1) Given the **partially ordered set**$<A,\preccurlyeq>$ as shown in the diagram, find the minimal elements, least element, maximal elements, and greatest element of *A*.

(2) Let $B=\{ b, c, d \}$, find the **lower bounds**, **upper bounds**, **infimum (greatest lower bound)**, and **supremum (least upper bound)** of *B*.




**Solution** (1)： Minimal elements: ***a*, *b*, *c***;

Maximal elements： ***a***， ***f***;

No least element or greatest element.

**Solution** (2) ：

**Lower bounds and greatest lower bound do not exist.**

**Upper bounds: d, f**

**Least upper bound (supremum): d**

- **Definition 4.29：** Let <$A$,≼>be a **partially ordered set (poset)**, and $B \subseteq A$.

(1) If for all $\forall x,y \in B$， ***x*** and ***y*** are **comparable**, then ***B*** is called a ***chain*** in *A*， The number of elements in ***B*** is called the **length of the chain.**

(2) If for all $\forall x,y \in B$， $x \neq y$， ***x*** and ***y*** are **not comparable**, then ***B*** is called an ***antichain*** in *A*， The number of elements in B is called the **length of the antichain..**

e.g. **Examples：** In the poset <{1,2,...,9},|>中， {1,2,4,8} is a **chain** of length 4， {1,4} is a **chain** of length 2， {2,3} is an **antichain** of length 2. The **singleton set** {2 has length 1 and is **both a chain and an antichain.**

# ↳ Antichain Decomposition Algorithm for Poset



- **Theorem 4.9：** Let **<$A$,≼>** be a **partially ordered set (poset)**. If the length of the longest chain in $A$ is $n$, then the poset can be decomposed into ***n*** ***disjoint antichains***.
- **Algorithm 4.2: Antichain Decomposition Algorithm for Posets.**

  **Input**: A partially ordered set $A$

  **Output**: Antichains $B_1$, $B_2$, …

  1. $i \leftarrow 1$
  2. $B_i \leftarrow$ the set of all **maximal elements** in A (which is an antichain)
  3. $A \leftarrow A - B_i$
  4. if $A \neq \varnothing$
  5.     $i \leftarrow i+1$
  6.     Go to 2

- **Topological Sorting:** Expanding a **partially ordered set (poset)** into a **totally ordered set** is called ***topological sorting***.
- **Algorithm 4.3: Topological Sorting**

**Input**: A partially ordered set $A$

**Output**: A sorted order of elements in $A$

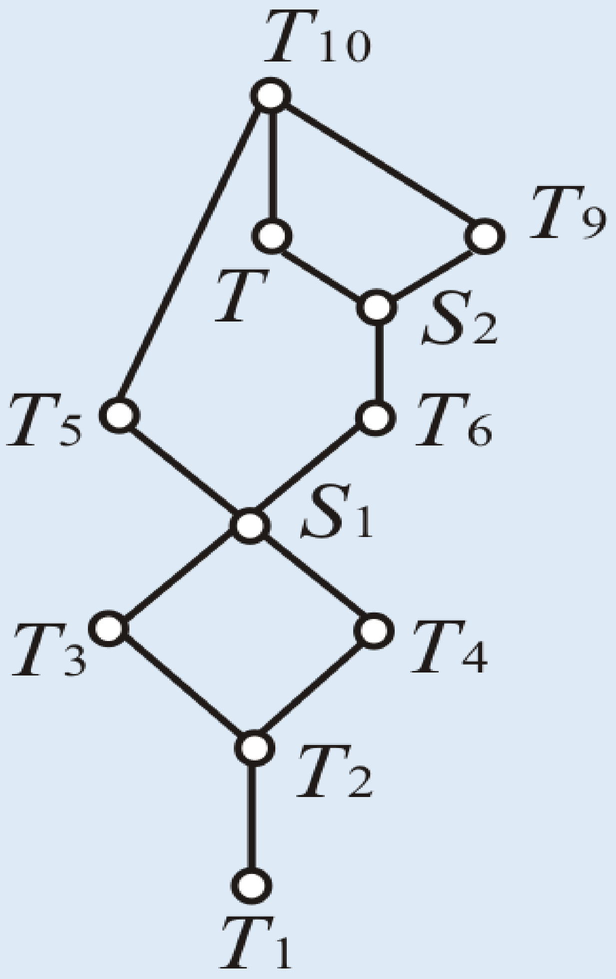
1. $i \leftarrow 1$
2. Select a **minimal element** $a_i$ from A and consider it the **smallest element**
3. $A \leftarrow A - \{a_i\}$
4. if $A \neq \varnothing$
5. $\quad i \leftarrow i+1$
6. $\quad$ Go to 2

## ↳ Topological Sorting - Extending a Poset to a Total Order(e.g.)



e.g. **Examle**: A set of tasks $A$ is given with partial order constraints, and its Hasse diagram is shown in the figure.

$A = \{T_1, T_2, T_3, T_4, T_5, S_1, T_6, S_2, T, T_9, T_{10}\}$




**Check whether the following topological order is valid.**

**such as:**

$T_1, T_2, T_3, T_4, S_1, T_5, T_6, S_2, T, T_9, T_{10};$

$T_1, T_2, T_3, T_4, S_1, T_6, S_2, T, T_9, T_5, T_{10};$

Objective :

Key Concepts ：